GENUS OF NUMERICAL SEMIGROUPS GENERATED BY THREE ELEMENTS

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Abstract. Let $H = \langle a, b, c\rangle$ be a numerical semigroup generated by three elements and let $R = k[H]$ be its semigroup ring over a field k. We assume H is not symmetric and assume that the definig ideal of R is defined by maximal minors of the matrix $\begin{pmatrix} X^\alpha & Y^\beta & Z^\gamma \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$. Then we will show that the genus of H is determined by the Frobenius number $\mathrm{F}(H)$ and $\alpha\beta\gamma$ or $\alpha'\beta'\gamma'$. In particular, we show that H is pseudo-symmetric if and only if $\alpha\beta\gamma = 1$ or $\alpha'\beta'\gamma' = 1$.

Also, we will give a simple algorithm to get all the pseudo-symmetric numerical semigroups $H = \langle a, b, c\rangle$ with give Frobenius number.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* H is a subset of $\mathbb{N}$ which is closed under addition and $\mathbb{N} \setminus H$ is a finite set. We always assume $0 \in H$.

We define $\mathrm{F}(H) := \max\{n \mid n \notin H\}$, and $\mathrm{g}(H) := \mathrm{Card}(\mathbb{N} \setminus H)$. We call $\mathrm{F}(H)$ the *Frobenius number* of H, and we call $\mathrm{g}(H)$ the *genus* of H. Then it is known that $2\,\mathrm{g}(H) \geq \mathrm{F}(H)+1$. We denote by $H = \langle a_1, a_2, ..., a_n\rangle$ the numerical semigroup generated by $a_1, a_2, ..., a_n$. Namely, $H = \sum_{i=1}^{n} a_i\mathbb{N}$. Moreover, every numerical semigroup admits a unique minimal system of generators.

We say that H is *symmetric* if $\mathrm{F}(H)$ is odd and for every $a \in \mathbb{Z}$, either $a \in H$ or $\mathrm{F}(H) - a \in H$, or equivalently, $2\,\mathrm{g}(H) = \mathrm{F}(H) + 1$. We say that H is *pseudo-symmetric* if $\mathrm{F}(H)$ is even and for every $a \in \mathbb{Z}, a \neq \mathrm{F}(H)/2$, either $a \in H$ or $\mathrm{F}(H) - a \in H$, or equivalently, $2\,\mathrm{g}(H) = \mathrm{F}(H) + 2$.

For a fixed field k, a variable t over k, let $R = k[H] = k[t^h \mid h \in H]$ be the semigroup ring of H. Then it is known that H semigroup is symmetric (resp. pseudo-symmetric) if and only $R = k[H]$ is a Gorenstein (resp. Kunz) ring (see [BDF]). The a-invariant of the semigroup ring R ([GW]) is defined to be $a(R) = \max\{n \mid [H^1_{\mathfrak{m}}(R)]_n \neq 0\}$. Since $H^1_{\mathfrak{m}}(R) \cong k[t, t^{-1}]/R$, $a(R) = \max\{m \mid m \notin H\}$, that is, $\mathrm{F}(H) = a(R)$.

We say that an integer x is a *pseudo-Frobenius number* of H if $x \notin H$ and $x+s \in H$ for all $s \in H, s \neq 0$. We denote by $\mathrm{PF}(H)$ the set of pseudo-Frobenius numbers of H. The cardinality in $\mathrm{PF}(H)$ is called the *type* of H, denoted by $\mathrm{t}(H)$. Since $x \in \mathrm{PF}(H)$ if and only if t^x is in the socle of $H^1_{\mathfrak{m}}(k[H])$, $\mathrm{t}(H) = \mathrm{r}(k[H])$, the Cohen-Macaulay type of $k[H]$. Since $\mathrm{F}(H) \in \mathrm{PF}(H)$, $\mathrm{t}(H) = 1$ if and only if H is symmetric.

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In this paper, we investigate numerical semigroups generated by three elements, which is not symmetric. We put $H = \langle a, b, c\rangle$ and always assume that H is *not* symmetric.

Let We now let $\varphi : S = k[X, Y, Z] \to R = k[H] = k[t^a, t^b, t^c]$ the k algebra homomorphism defined by $\varphi(X) = t^a$, $\varphi(Y) = t^b$, and $\varphi(Z) = t^c$ and let $\mathfrak{p} = \mathfrak{p}(a, b, c)$ be the kernel of φ. Then it is known that if H is not symmetric, then the ideal $\mathfrak{p} = \mathrm{Ker}(\varphi)$ is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix} \tag{1.1}$$

for some positive integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ (cf. [He]). We want to describe $\mathrm{g}(H)$ by $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ and the main goal of this paper is the following Theorem.

Theorem. Let H be a numerical semigroup as above. Then
(1) if $\beta' b > \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$,
(2) if $\beta' b < \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha'\beta'\gamma'$.

As a direct consequence of this Theorem, we can get the characterization of pseudo-symmetric semigroups generated by 3 elements.

Corollary. Let H be a numerical semigroup as above. Then H is pseudo-symmetric if and only if either $\alpha = \beta, = \gamma = 1$ or $\alpha' = \beta' = \gamma' = 1$.

Also, we will give an algorithm to classify all pseudo-symmetric numerical semigroup H generated by 3 elements with given Frobenius number $\mathrm{F}(H)$.

2. Numerical semigroups generated by three elements

Let $H = \langle a, b, c\rangle$ be a numerical semigroup and $R = k[H] \cong k[X, Y, Z]/\mathfrak{p}$ be its semigroup ring over a field k. Then it is known that the ideal $\mathfrak{p}$ of $S = k[X, Y, Z]$ is generated by the maximal minors of the matrix $\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$, where α, β, γ, α', β', and γ' are positive integers. Since $k[H]/(t^a) \cong k[Y, Z]/(Y^{\beta+\beta'}, Y^{\beta'}Z^{\gamma}, Z^{\gamma+\gamma'})$, the defining ideal of $k[H]/(t^a)$ is generated by the maximal minors of the matrix $\begin{pmatrix} 0 & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & 0 \end{pmatrix}$. Since $a = \dim_k k[H]/(t^a) = \dim_k k[Y, Z]/(Y^{\beta+\beta'}, Y^{\beta'}Z^{\gamma}, Z^{\gamma+\gamma'})$, and likewise for b, c, we get the equalitions

$$\begin{array}{rcl} a & = & \beta\gamma + \beta'\gamma + \beta'\gamma', \\ b & = & \gamma\alpha + \gamma'\alpha + \gamma'\alpha', \\ c & = & \alpha\beta + \alpha'\beta + \alpha'\beta'. \end{array} \tag{2.1.1}$$

We put $l = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta}$, $m = X^{\alpha+\alpha'} - Y^{\beta'}Z^{\gamma}$, and $n = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma'}$. There are obvious relations

$$X^{\alpha}l + Y^{\beta}m + Z^{\gamma}n = Y^{\beta'}l + Z^{\gamma'}m + X^{\alpha'}n = 0.$$

We put $p = \deg(Z^{\gamma+\gamma'})$, $q = \deg(X^{\alpha+\alpha'})$, $r = \deg(Y^{\beta+\beta'})$, $s = \deg(X^{\alpha}) + p$, $t = \deg(Y^{\beta'}) + p$. Since $\mathrm{pd}_S(R) = 2$, we get a free resolution of R

$$0 \to S(-s) \oplus S(-t) \to S(-p) \oplus S(-q) \oplus S(-r) \to S \to R \to 0.$$

Taking $\mathrm{Hom}_S(*, K_S) = \mathrm{Hom}_S(*, S(-x))$, we get

$$0 \to S(-x) \to S(p-x) \oplus S(q-x) \oplus S(r-x) \to S(s-x) \oplus S(t-x) \to K_R \to 0,$$

where $x = a+b+c$ and $K_R = \mathrm{Ext}^2_S(R, K_S)$.

Since K_R is generated by the elements of degree $-\mathrm{PF}(H)$, from this exact sequence, we have that $\mathrm{PF}(H) = \{s-x, t-x\}$. We put $f = s-x$ and $f' = t-x$.

By the above argument, we obtain the following results.

Proposition 2.1. *If $H = \langle a, b, c\rangle$ is not symmetric, then*

1. $(\alpha + \alpha')a = \beta' b + \gamma c$ *and* $\alpha + \alpha' = \min\{n \mid an \in \langle b, c\rangle\}$,
2. $(\beta + \beta')b = \alpha a + \gamma' c$ *and* $\beta + \beta' = \min\{n \mid bn \in \langle a, c\rangle\}$,
3. $(\gamma + \gamma')c = \alpha' a + \beta b$ *and* $\gamma + \gamma' = \min\{n \mid cn \in \langle a, b\rangle\}$.

Proposition 2.2. *If $H = \langle a, b, c\rangle$ is not symmetric, then $\mathrm{PF}(H) = \{f, f'\}$ where*

1. $f = \alpha a + (\gamma + \gamma')c - (a+b+c)$,
2. $f' = \beta' b + (\gamma + \gamma')c - (a+b+c)$.

Remark 2.3. Formulas related to our results in this section can be found in [RG1], [RG].

3. Main results

The following is the key lemma to prove our main theorem.

Lemma 3.1. *Let $H = \langle a, b, c\rangle$ be as in the previous section. We assume that $\beta' b > \alpha a$, or equivalently, $f' > f$. Then*

1. *for $p, q, r \in \mathbb{N}$, $f' - f + pa + qb + rc \notin H$ if and only if $p < \alpha, q < \beta$ and $r < \gamma$.*
2. $\mathrm{Card}\{h \in H \mid f' - f + h \notin H\} = \alpha\beta\gamma$.
3. $\mathrm{Card}[[(f - H) \cap \mathbb{N}] \setminus (f' - H)] = \alpha\beta\gamma$.

Proof. Since $f' - f + \alpha a = b\gamma, f' - f + \beta b = \gamma' c, f' - f + \gamma c = \alpha' a \in H$, $f' - f + pa + qb + rc \in H$ if $p \geq \alpha$ or $q \geq \beta$ or $r \geq \gamma$. Conversely, assume $p < \alpha, q < \beta$ and $r < \gamma$ and $f' - f + pa + qb + rc = ua + vb + wc \in H$ for some $u, v, w \in \mathbb{N}$. Then we have $(\beta' + q - v)b = (\alpha - p + u)a + (v - r)c$. If $v \geq r$, then this contradicts Proposition 2.1 (2). If $r > v$, we have $(\alpha - p + u)a = (\beta' + q - v)b + (r - v)c$. Then by Proposition 2.1 (1), we must have $p - u \geq \alpha'$ and again we have a contradiction since $r - v < \gamma$. This finishes the proof of (1) and (2) is a direct consequence of (1).

To show (3), it suffices to note that for $h \in H$, $f - h \notin f' - H$ if and only if $f' - (f - h) \notin H$.

Thus we have $\mathrm{Card}[(f - H) \setminus (f' - H)] = \mathrm{Card}\{h \in H \mid f' - f + h \notin H\} = \alpha\beta\gamma$. $\square$

Theorem 3.2. *Let $H = \langle a, b, c\rangle$ be a numerical semigroup. Then*

1. *if $\beta' b > \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha\beta\gamma$,*
2. *if $\beta' b < \alpha a$, then $2\,\mathrm{g}(H) - (\mathrm{F}(H) + 1) = \alpha'\beta'\gamma'$.*

Proof. We may assume $\beta' b > \alpha a$. Then by Proposition 2.2, $\mathrm{F}(H) = f'$. Since $\mathbb{N} \setminus H = ((f' - H) \cap \mathbb{N}) \cup ((f - H) \cap \mathbb{N})$, we get

$$g(H) = \mathrm{Card}[(f' - H) \cap \mathbb{N}] + \mathrm{Card}[[(f - H) \cap \mathbb{N}] \setminus (f' - H)]$$

hence by Lemma 3.1,

$$g(H) = (F(H) + 1 - g(H)) + \alpha\beta\gamma.$$

□

As a corollary, we find a characterization of pseudo-symmetric numerical semigroups generated by 3 elements.

Corollary 3.3. *H is pseudo symmetric if and only if*

(1) *if $\beta' b > \alpha a$, then $\alpha = \beta = \gamma = 1$ and*
(2) *if $\beta' b < \alpha a$, then $\alpha' = \beta' = \gamma' = 1$.*

Proof. We may assume that $\beta' b > \alpha a$. By Theorem 3.2, $2\,\mathrm{g}(H) - (\mathrm{F}(H)+1) = \alpha\beta\gamma$. Since H is pseudo-symmetric if and only if $2\,\mathrm{g}(H) = \mathrm{F}(H) + 2$, we obtain that $\alpha\beta\gamma = 1$, or equivalently, $\alpha = \beta = \gamma = 1$. □

4. The structure of a pseudo-symmetric numerical semigroup generated by three elements

In this section, we assume that $H = \langle a, b, c\rangle$ is a pseudo-symmetric numerical semigroup. Our purpose is to classify, for any fixed fixed even integer f, all the pseudo-symmetric numerical semigroups $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = f$. For example, it is shown in Exercise 10.8 of [RG] that there is no pseudo-symmetric numerical semigroup $H = \langle a, b, c\rangle$ with $F(H) = 12$. Actually, we can give many examples of such even integer f for which there does not exist numerical semigroup $H = \langle a, b, c\rangle$ with $F(H) = f$. (It is shown in [RGG] that every even integer is the Frobenius number of some numerical semigroup generated by at most 4 elements.)

As is mentioned before, $\mathfrak{p} = \mathfrak{p}(a, b, c)$ of $k[X, Y, Z]$ is generated by the maximal minors of the matrix as in (1.1) and by Corollary 3.3, we can always assume that $\alpha' = \beta' = \gamma' = 1$. Recall that in this case we have by (2.1.1),

$$a = \beta\gamma + \gamma + 1, \quad b = \gamma\alpha + \alpha + 1, \quad c = \alpha\beta + \beta + 1. \tag{4.1.1}$$

The following is the key for our goal.

Theorem 4.1. *Let $H = \langle a, b, c\rangle$ be a pseudo-symmetric numerical semigroup and assume that $\mathfrak{p}(a, b, c)$ is generated by the maximal minors of the matrix $\begin{pmatrix} X^\alpha & Y^\beta & Z^\gamma \\ Y & Z & X \end{pmatrix}$. Then we have*

$$\alpha\beta\gamma = \frac{\mathrm{F}(H)}{2} + 1.$$

Proof. From our hypothesis and Corollary 3.3, we have $f' < f$. Thus by Proposition 2.2, $\mathrm{F}(H) = f = \alpha a + (\gamma + 1)c - (a + b + c) = 2\alpha\beta\gamma - 2$. □

Now, given a positive even integer f, we can list all possibilities of the set $\{\alpha, \beta, \gamma\}$ by prime factorization of $\frac{\mathrm{F}(H)}{2} + 1$.

Remark 4.2. Let σ be a permutation of $\{\alpha, \beta, \gamma\}$. Then it is easy to see that if σ is an even permutation, then the set $\{a, b, c\}$ obtained by $\{\sigma(\alpha), \sigma(\beta), \sigma(\gamma)\}$ as in (4.1.1) is the same and hence the semigroup $H = \langle a, b, c\rangle$ does not change.

But if σ is an odd permutation, then the set $\{a, b, c\}$ does change. So, from the factorization of $\dfrac{\mathrm{F}(H)}{2} + 1$, we get 2 different semigroups in general.

Example 4.3. For example, let us classify all pseudo-symmetric semigroup $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = f = 18$. Since we have $\alpha\beta\gamma = f/2 + 1 = 10$ by Theorem 4.1, we have $\{\alpha, \beta, \gamma\} = \{10, 1, 1\}$ or $\{5, 2, 1\}$. But if we put $\{\alpha, \beta, \gamma\} = \{10, 1, 1\}$ in any order to (4.1.1), a, b, c are all multiple of 3 and we don't get a numerical semigroup.

Thus we get 2 semigroups with $\mathrm{F}(H) = 18$; if $(\alpha, \beta, \gamma) = (5, 2, 1)$ we get $H = \langle 4, 11, 13\rangle$ and if $(\alpha, \beta, \gamma) = (5, 1, 2)$, then we get $H = \langle 5, 16, 7\rangle$.

If f is an even integer not divisible by 12, then there is a pseudo-symmetric semigroup $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = f$ by [RGG].

Proposition 4.4. [RGG] *Let $H = \langle a, b, c\rangle$ be a numerical semigroup and $\mathrm{F}(H) = f$. Then*

(1) *If f is an even integer not divisible by 3, then*

$$\left\langle 3, \frac{f}{2} + 3, f + 3 \right\rangle$$

is a pseudo-symmetric numerical semigroup with Frobenius number f. We put $(\alpha, \beta, \gamma) = (f/2 + 1, 1, 1)$.

(2) *If f is a multiple of 6 and not a multiple of 12, then if we put $(\alpha, \beta, \gamma) = ((f + 2)/4, 2, 1)$, we get*

$$H = \left\langle 4, \frac{f}{2} + 2, \frac{f}{2} + 4 \right\rangle,$$

which is pseudo-symmetric with $\mathrm{F}(H) = f$.

If f is divisible by 12, there are many cases such that there does not exist pseudo-symmetric semigroup $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = f$.

Proposition 4.5. *We suppose $12 \mid f$. If there exists a pseudo-symmetric numerical semigroup $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = f$, then $f/2 + 1$ has a prime factor of the form $3k + 2$ $(k \geq 1)$.*

Proof. Otherwise, since α, β, γ are divisors of $f/2 + 1$, we get $\alpha \equiv \beta \equiv \gamma \equiv 1 \pmod 3$. Then by (4.1.1) we see that a, b, c are divisible by 3 and $H = \langle a, b, c\rangle$ is not a numerical semigroup. $\square$

Example 4.6. Let f be an integer divisible by 12.

(1) By Proposition 4.5, there is no pseudo-symmetric semigroup $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = 12, 24, 36, 60, 72, 84, 96, 120, 132, 144, 156, 180, 192$.

(2) On the other hand, there exists pseudo-symmetric semigroups $H = \langle a, b, c\rangle$ with $\mathrm{F}(H) = 48, 168$. Actually, $H = \langle 7, 11, 31\rangle$ is the unique pseudo-symmetric semigroup generated by 3 elements, with $\mathrm{F}(H) = 48$ and $H = \langle 19, 11, 103\rangle$ is the unique pseudo-symmetric semigroup generated by 3 elements with $\mathrm{F}(H) = 168$

(3) The converse of Proposition 4.5 is not true. Indeed, If $f = 1596$, then $f/2 + 1 = 799 = 17 \times 47$ has a prime factor which is congruent to 2 mod 3. But if we substitute $(\alpha, \beta, \gamma) = (17, 47, 1)$ (resp. $(47, 17, 1)$) in (4.1.1), then we get $(a, b, c) = (49, 35, 847)$ (resp. $(19, 95, 817)$). These are not numerical semigroups since (a, b, c) have common prime factor. It is not difficult to show that $f = 1596$ is the smallest of such examples.

5. Simple numerical semigroups

Let H be a numerical semigroup with minimal system of generators $\{a_1, a_2, ..., a_n\}$. We assume that a_1 is the least positive integer in H. For every $i \in \{1, ..., n\}$, set

$$\delta_i := \min\{k \in \mathbb{N} \setminus \{0\} \mid ka_i \in \langle \{a_1, ..., a_n\} \setminus \{a_i\} \rangle \}.$$

The notion of simple numerical semigroup was defined in Exercise 10.3 of [RG].

Definition 5.1. We say that H is *simple* if $a_1 = (\delta_2 - 1) + (\delta_3 - 1) + \cdots + (\delta_n - 1) + 1$.

Proposition 5.2. *Let $H = \langle a_1, a_2, ..., a_n \rangle$ be a simple numerical semigroup. Then the type of H is $n - 1$. Hence if H is simple with $n \geq 3$, then H is not symmetric.*

Proof. By definition of pseudo-Frobenius number, we have that

$$\mathrm{PF}(H) = \{(\delta_2 - 1)a_2 - a_1, (\delta_3 - 1)a_3 - a_1, ..., (\delta_n - 1)a_n - a_1\},$$

that is, H has type $n - 1$. □

The following is the main result in this section.

Theorem 5.3. *Let $H = \langle a, b, c \rangle$ be a numerical semigroup defined by the matrix as in (1.1). If we assume that a is the least positive integer in H, then H is simple if and only if $\beta' = \gamma = 1$.*

Proof. Since $a = \beta\gamma + \beta'\gamma + \beta'\gamma'$, and since we have $\delta_2 = \beta + \beta', \delta_3 = \gamma + \gamma$, H is simple if and only if

$$\beta\gamma + \beta'\gamma + \beta'\gamma' = \beta + \beta' + \gamma + \gamma' - 1$$

or, equivalently,

$$(\beta - 1)(\gamma - 1) + (\beta' - 1)(\gamma' - 1) + (\beta'\gamma - 1) = 0.$$

Since $\beta, \beta', \gamma, \gamma'$ are positive integers, the latter equation is equivalent to $\beta' = \gamma = 1$. □

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